§ Totally Umbilic Surfaces

Thm: $S \subseteq \mathbb{R}^3$ connected totally umbilic $\Longrightarrow$ S is contained in a plane or sphere.

Proof: Recall that totally umbilic means

$$\kappa_1(p) = \kappa_2(p) \quad \text{at every } p \in S$$

i.e. $\exists$ function $f : S \to \mathbb{R}$ s.t.

$$S = -dN_p = f(p)\, \mathrm{Id} : T_pS \to T_pS$$

Ex: Show that f is smooth!

For any parametrization $X(u,v)$ on S,

$$\begin{cases} S\left(\dfrac{\partial X}{\partial u}\right) = f \dfrac{\partial X}{\partial u} \\ S\left(\dfrac{\partial X}{\partial v}\right) = f \dfrac{\partial X}{\partial v} \end{cases} \Rightarrow \begin{cases} -\dfrac{\partial N}{\partial u} = f \dfrac{\partial X}{\partial u} \\ -\dfrac{\partial N}{\partial v} = f \dfrac{\partial X}{\partial v} \end{cases} \quad (*)$$

$$\Rightarrow \begin{cases} -\dfrac{\partial^2 N}{\partial v \partial u} = \dfrac{\partial f}{\partial v}\dfrac{\partial X}{\partial u} + f \dfrac{\partial^2 X}{\partial v \partial u} \\ -\dfrac{\partial^2 N}{\partial u \partial v} = \dfrac{\partial f}{\partial u}\dfrac{\partial X}{\partial v} + f \dfrac{\partial^2 X}{\partial u \partial v} \end{cases}$$

$$\Rightarrow \quad \frac{\partial f}{\partial v}\frac{\partial X}{\partial u} = \frac{\partial f}{\partial u}\frac{\partial X}{\partial v}$$

$$\Rightarrow \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial u} \equiv 0 \quad \left(\because \left\{\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\right\} \text{ lin. indep.}\right)$$

i.e. f is (locally) constant ($\because S$ connected)

<u>Case 1</u>: $\boxed{f \equiv 0}$ $\Rightarrow$ $N \equiv \text{const.}$ plane!

<u>Case 2</u>: $\boxed{f \equiv c \neq 0}$

<u>Claim</u>: S is contained in a sphere of radius $\frac{1}{|c|}$

It suffices to show:

$$\mathbf{X} + \frac{1}{f} N \equiv \text{const. } p_0$$ ← center of the sphere

Note that:

$$\frac{\partial}{\partial u}\left(\mathbf{X} + \frac{1}{f} N\right) = \frac{\partial \mathbf{X}}{\partial u} + \frac{1}{f}\frac{\partial N}{\partial u} \overset{(*)}{\equiv} 0 .$$

Similarly,

$$\frac{\partial}{\partial u}\left(\mathbf{X} + \frac{1}{f} N\right) = 0 .$$

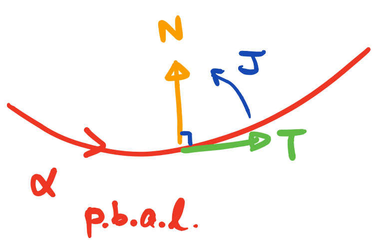
This proves the claim since S is connected.

$\square$

§ Normal curvatures

We now want to interpret the 2nd f.f. A as evaluating the curvature of certain plane curves lying on S.

<u>Recall</u>:



$$\boxed{k = \langle \alpha'', N \rangle}$$

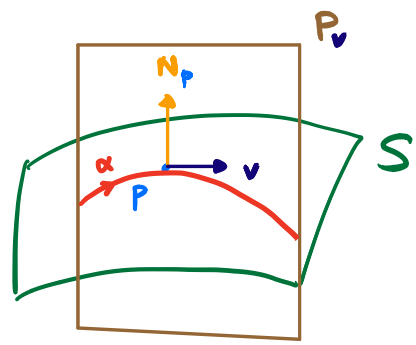
with orientation $\{T, N\}$

Let $S \subseteq \mathbb{R}^3$ be a surface oriented by N.

$\boxed{\text{Fix } p \in S \text{ and a unit tangent vector } v \in T_pS}$

Consider the oriented plane

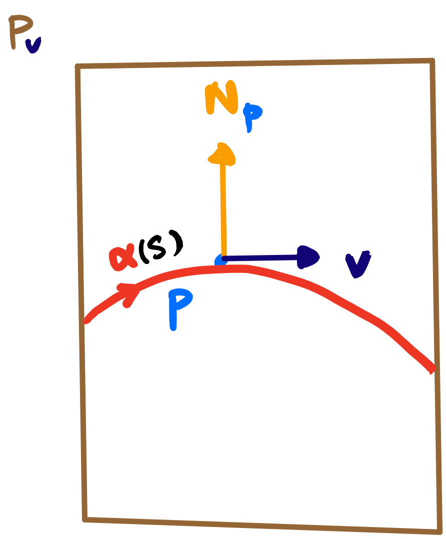
$$P_v = \text{span}\{\underbrace{v, N_p}_{\text{pos. orientation}}\}$$



which cuts S along some regular curve (why?) p.b.a.l.

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow S \quad \text{s.t.} \quad \alpha(0) = p \ , \quad \alpha'(0) = v$$

which can also be regarded as a plane curve on P_v



with curvature

$$\boxed{k_v = \langle \alpha''(0), N_p \rangle} \quad \text{— } (*)$$

On the other hand, since $\alpha \subseteq S$

$$\Longrightarrow \quad \alpha'(s) \in T_{\alpha(s)}S \ , \ \forall s$$

$$\Longrightarrow \quad \langle \alpha'(s), N(\alpha(s)) \rangle \equiv 0 \quad \forall s$$

Differentiate w.r.t. s at $s=0$

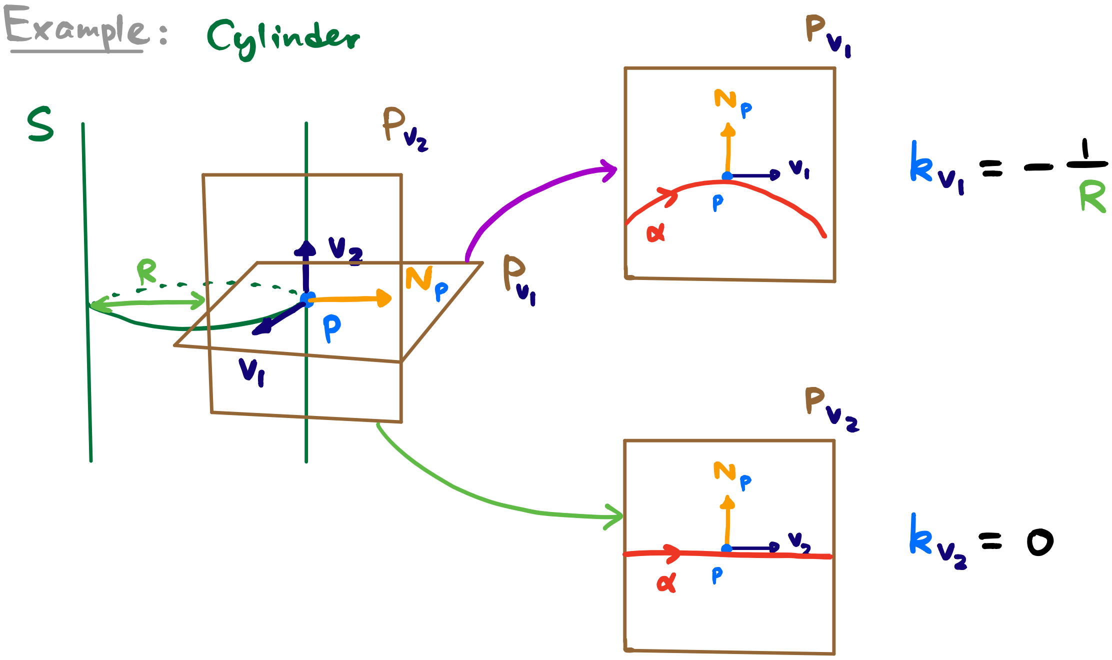
$$\Longrightarrow \underbrace{\langle \alpha''(0), N_p \rangle}_{k_v} + \underbrace{\langle \alpha'(0), dN_p(\alpha'(0)) \rangle}_{-A(v,v)} = 0$$

i.e. $A(v,v) = k_v$ (normal curvature along v)

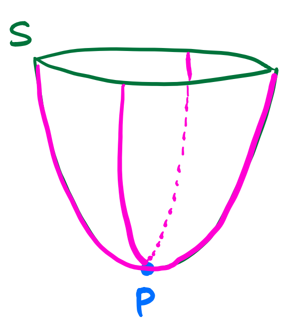
By the variational characterization of eigenvalues, the principal curvatures (at p) are

$$\boxed{\kappa_1 = \min_{\substack{v \in T_pS \\ \|v\|=1}} k_v} \quad \& \quad \boxed{\kappa_2 = \max_{\substack{v \in T_pS \\ \|v\|=1}} k_v}$$

Example: Cylinder

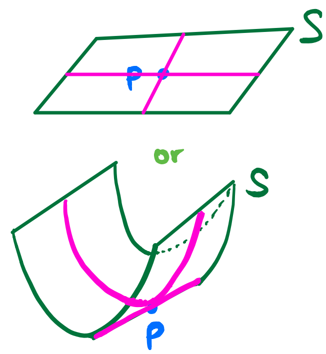


We have the following local picture of surfaces:



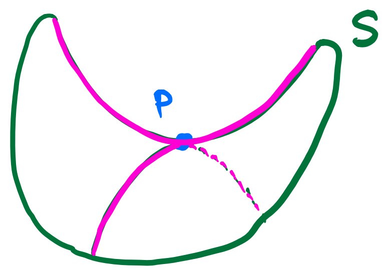
$K > 0$

"elliptic"



$K = 0$

"planar / parabolic"



$K < 0$

"hyperbolic"